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# A SOLUTION TO THE DISTRIBUTION PROBLEMS ARISING IN THE STUDIES OF A TWO-SEX POPULATION PROCESS

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Given a population of two sexes, the birth rate of one sex of which depends upon the population size of the other, it is very difficult to find an explicit expression for the probability distribution of the former. In this paper we have explicitly found the probability generating function of the joint distribution from which individual probability distributions and, in particular, moments of all orders in each case can be obtained in principle. As an example, using this probability generating function we have worked out explicitly the first and second order moments of the male and female populations and the explicit expression for the distribution of the male population in a particular case. This method can be successfully applied for the same purpose in the studies of chemical and biological processes where the synthesis or production of one species depends upon the concentration of another species.

marriage dominance * marginal distribution * probability generating function * Riccati equation * confluent hypergeometric function

## 1. Introduction

The stochastic process describing a population of two sexes where the females are marriage dominant i.e., where the birth rates of both males and females depend only on the female population size has been dealt with by Goodman [4], Joshi [5], Bharucha-Reid [2], Bailey [1], Pollard [7] and others.

The stochastic process for the growth of population under consideration gives rise to a bivariate process of the birth-and-death type. Since the female subpopulation which has been supposed to be marriage dominant follows a simple birth-and-death process, the probability distribution of the females in the population can be explicitly obtained but the behaviour of the male population size (which is not marriage dominant) is somewhat more difficult to analyse since male births depend on the female population. In fact no direct method for obtaining an explicit expression for the distribution of males in such a population has yet been found. However, the above authors have utilized the method of moment or cumulant generating functions to obtain the first and second order moments of the males and females. But this method turns out to be unwieldy when attempting to obtain higher order moments. Furthermore their studies do not shed any light on the nature of the probability distribution of the male population which is definitely of much interest. In the cases where neither males nor females are marriage dominant i.e., where the contribution from the birth rates of both the subpopulations depend on the total population size, neither of the distributions of males or females follow a simple birth-and-death process and thus it is difficult to obtain an expression of the distribution of each of the sexes in such cases. In this paper we have explicitly found the probability generating function of the joint probability distribution in such cases i.e., when either or both of the sexes are not marriage dominant by using standard mathematical techniques of solving nonlinear differential equations. Meanwhile it is brought to our notice [3b] that Gani and Tin [3a] have simultaneously obtained a solution of similar type by a different approach, but concentrating rather on the structure of the process. In our case, we have concentrated on the analysis of the relevant joint and marginal distributions.

## 2. The stochastic model of population growth

Much of this section is standard birth-and-death process material, derived initially by Goodman [4] and referred to in various texts ([1, 2, 7]). The results are briefly discussed here.

If $X(t)$ and $Y(t)$ denote the numbers of females and males in the population at time $t$ and if

$$P_{x,y}(t) = \mathcal{P}\{X(t)=x,\ Y(t)=y\}, \quad x \geq 0, y \geq 0$$

denotes the joint probability distribution of $X(t)$ and $Y(t)$, then the population may undergo the following transitions:

| Transition | Probability in unit time |
|---|---|
| $x, y \to (x+1, y)$ | $\lambda p x$ |
| $\to (x, y+1)$ | $\lambda q x$ |
| $\to (x-1, y)$ | $\mu x$ |
| $\to (x, y-1)$ | $\mu' y$ |
| $\to (x, y)$ (i.e., no change) | $1-[\lambda(p+q)+\mu x+\mu' y]$ |

where $\lambda > 0$, $\mu \geq 0$, $\mu' \geq 0$ and $p+q=1$.

Hence it is clear that the probabilities $P_{x,y}(t)$ satisfy the following system of differential-difference equations:

$$\frac{\mathrm{d}P_{x,y}(t)}{\mathrm{d}t} = -[(\lambda+\mu)x+\mu' y]P_{x,y}(t)+[\lambda p(x-1)]P_{x-1,y}(t)$$
$$+\lambda q x P_{x,y-1}(t)+\mu(x+1)P_{x+1,y}(t)+\mu'(y+1)P_{x,y+1}(t) \tag{2.1}$$
$$(\text{for } x, y = 0, 1, 2, \ldots),$$

$$
\begin{aligned}
P_{x,y}(0) &= 1 \quad \text{for } (x, y) = (k_1, k_2), \\
&= 0 \quad \text{for } (x, y) \neq (k_1, k_2).
\end{aligned}
\tag{2.2}
$$

and $P_{x,y}(t) = 0$ for $x, y < 0$.

By appropriate summations, one can obtain from equation (2.1)

$$\frac{\mathrm{d}E[X(t)]}{\mathrm{d}t} = \lambda p E[X(t)] - \mu E[X(t)], \tag{2.3}$$

$$\frac{\mathrm{d}E[Y(t)]}{\mathrm{d}t} = \lambda q E[X(t)] - \mu' E[Y(t)], \tag{2.4}$$

which on solving gives

$$E[X(t)] = k_1 \, \mathrm{e}^{(\lambda p - \mu)t}, \tag{2.5}$$

$$E[Y(t)] = k_2 \, \mathrm{e}^{-\mu' t} + \frac{\lambda q k_1}{\lambda p - \mu + \mu'} [\mathrm{e}^{(\lambda p - \mu)t} - \mathrm{e}^{-\mu' t}] \tag{2.6}$$

where $E[X(0)] = k_1$ and $E[Y(0)] = k_2$ are the initial values. From the expression for the expected number of the female sub-population $X(t)$ (equation (2.5)) and from the assumption that the probability of transition of $X(t)$ from state $x$ to state $x+1$ or $x-1$ in $(t, t+\Delta t)$ is proportional to $x$, the constant of proportionality being $\lambda p$ or $\mu$ respectively, it is clear that the growth of the females follows a simple birth-and-death process. By the standard method [1, 2, 7] the marginal distribution of $X(t)$ when $k_1 = 1$ can easily be found to be, for $\lambda p \neq \mu$,

$$P_x(t) = [1 - P_0(t)] \left[ 1 - \frac{\lambda p}{\mu} P_0(t) \right] \left( \frac{\lambda p}{\mu} P_0(t) \right)^{x-1}, \quad x = 1, 2, 3, \ldots, \tag{2.7}$$

where

$$P_0(t) = \frac{\mu (\mathrm{e}^{(\lambda p - \mu)t} - 1)}{\lambda p \, \mathrm{e}^{(\lambda p - \mu)t} - \mu},$$

with probability generating function (p.g.f.)

$$F(r, t) = \frac{\mu (1 - \mathrm{e}^{-(\lambda p - \mu)t}) + r(\lambda p \, \mathrm{e}^{-(\lambda p - \mu)t} - \mu)}{\lambda p - \mu \, \mathrm{e}^{-(\lambda p - \mu)t} - r \lambda p (1 - \mathrm{e}^{-(\lambda p - \mu)t})}, \quad |r| \leq 1, \tag{2.8}$$

and when $\lambda p = \mu$, then

$$P_x(t) = (1 - P_0)^2 P_0^{x-1}, \quad x = 1, 2, 3, \ldots. \tag{2.9}$$

where

$$P_0(t) = \frac{\lambda p t}{1 + \lambda p t}$$

with the p.g.f. which is now

$$F(r, t)=\frac{\lambda pt+r(1-\lambda pt)}{1+\lambda pt-r\lambda pt}, \quad |r|\leqslant 1. \tag{2.10}$$

The growth of the male subpopulation $Y(t)$ does not follow a simple birth-and-death process, since the transition probability from state $y$ to state $y+1$ in $(t, t+\Delta t)$ is proportional to $x(t)$ (the number of females in the population at time $t$). Hence the marginal distribution $P_y(t)$ cannot be obtained by the previous method.

Thus in order to have an explicit expression for the distribution $P_y(t)$ also, we go back to the original master equation (2.1) and make attempts to solve this equation for the generating function of the joint probability distribution $P_{x,y}(t)$ using standard mathematical techniques of solving nonlinear differential equations.

## 3. Solution of the master equation (2.1) in terms of the probability generating function (p.g.f.)

Let

$$F(r, s, t)=\sum_{x=0}^{\infty}\sum_{y=0}^{\infty} P_{x,y}(t)r^x s^y, \quad |r|\leqslant 1, \ |s|\leqslant 1 \tag{3.1}$$

be the generating function of the probabilities $P_{x,y}(t)$. Then, from equation (2.1), $F(r, s, t)$ satisfies the partial differential equation

$$\frac{\partial F}{\partial t}=(\lambda pr^2+\lambda qrs-\lambda r-\mu r+\mu)\frac{\partial F}{\partial r}+\mu'(1-s)\frac{\partial F}{\partial s}. \tag{3.2}$$

The subsidiary equations are

$$\frac{\mathrm{d}r}{\lambda pr^2+\lambda qrs-\lambda r-\mu r+\mu}=\frac{\mathrm{d}s}{\mu'(1-s)}=\frac{-\mathrm{d}t}{1}=\frac{\mathrm{d}F}{0}. \tag{3.3}$$

The last and third items of (3.3) give the integral

$$F=\text{Constant}, \tag{3.4}$$

while the second and third give the integral

$$(s-1)\,\mathrm{e}^{-\mu' t}=\text{Constant}=c_1 \quad \text{(say)} \tag{3.5}$$

and the first and third items give the equation

$$\mathrm{d}r=-[\lambda pr^2+\lambda qrs-(\lambda+\mu)r+\mu]\,\mathrm{d}t. \tag{3.6}$$

Substituting $s$ from (3.5) and then putting $\mathrm{e}^{\mu' t}=\bar{x}$ in (3.6) gives

$$-\frac{\mathrm{d}r}{\mathrm{d}t}=\frac{\lambda pr^2+\lambda qr+\lambda qrc_1\bar{x}-(\lambda+\mu)r+\mu}{\mu'\bar{x}}. \tag{3.7}$$

Now, making the transformation $r=(\bar{y}+1)$, (3.7) reduces to

$$\frac{\mathrm{d}\bar{y}}{\mathrm{d}x}=-\frac{\lambda p\bar{y}^2}{\mu'\bar{x}}-\frac{(\lambda p+\lambda qc_1\bar{x}-\mu)\bar{y}}{\mu'\bar{x}}-\frac{\lambda qc_1}{\mu'}. \tag{3.8}$$

Equation (3.8) can be written as

$$\bar{y}'=f(\bar{x})+g(\bar{x})\bar{y}+h(\bar{x})\bar{y}^2 \tag{3.9}$$

which is a generalized Riccati equation where

$$f(\bar{x})=-\frac{\lambda qc_1}{\mu'},\quad g(\bar{x})=-\frac{\lambda p+\lambda qc_1\bar{x}-\mu}{\mu'\bar{x}}\quad\text{and}\quad h(\bar{x})=-\frac{\lambda p}{\mu'\bar{x}}. \tag{3.10}$$

We give the transformation to the Riccati equation (3.9)

$$yh(\bar{x})u(\bar{x})+u'(\bar{x})=0,\qquad \log u(\bar{x})+yh(\bar{x})\,\mathrm{d}\bar{x}=0. \tag{3.11}$$

The result is

$$u''(\bar{x})+P(\bar{x})u'(\bar{x})+Q(\bar{x})u(\bar{x})=0 \tag{3.12}$$

where $P(\bar{x})=-(g+h'/h)$, $Q(\bar{x})=f(\bar{x})h(\bar{x})$, i.e.,

$$P(\bar{x})=\frac{\lambda p-\mu+\lambda qc_1\bar{x}+\mu'}{\mu'\bar{x}}\quad\text{and}\quad Q(\bar{x})=\frac{\lambda^2pqc_1}{\mu'^2\bar{x}}\quad(\text{from (3.10)}). \tag{3.13}$$

Substituting (3.13) in (3.12) gives

$$u''(\bar{x})+\frac{\lambda p-\mu+\lambda qc_1\bar{x}+\mu'}{\mu'\bar{x}}u'(\bar{x})+\frac{\lambda^2pqc_1}{\mu'^2\bar{x}}u(\bar{x})=0 \tag{3.14}$$

which can be rearranged as

$$\bar{x}u''(\bar{x})+\left(\frac{\lambda p-\mu+\mu'}{\mu'}+\frac{\lambda qc_1\bar{x}}{\mu'}\right)u'(\bar{x})+\frac{\lambda^2pqc_1}{\mu'^2}u(\bar{x})=0. \tag{3.15}$$

Let $\bar{x}=\beta z$. Then (3.15) becomes

$$zu''(z)+\left(\frac{\lambda p-\mu+\mu'}{\mu'}+\frac{\lambda qc_1\beta z}{\mu'}\right)u'(z)+\frac{\lambda^2pqc_1\beta}{\mu'^2}u(z)=0. \tag{3.16}$$

Putting

$$-\frac{\lambda qc_1\beta}{\mu'}=1,\quad\text{i.e., }\beta=-\frac{\mu'}{\lambda qc_1},$$

equation (3.16) can be written as

$$zu''(z)+\left(\frac{\lambda p-\mu+\mu'}{\mu'}-z\right)u'(z)-\frac{\lambda p}{\mu'}u(z)=0. \tag{3.17}$$

This is a confluent hypergeometric equation (Kumer's equation) having a regular singularity at $z=0$ and an irregular singularity at $\infty$. Its independent solutions are (writing in terms of $\bar{x}$):

$$\begin{aligned} u_1(\bar{x}) &= M\left(\frac{\lambda p}{\mu'}, \frac{\lambda p-\mu+\mu'}{\mu'}, -\frac{\lambda q c_1 \bar{x}}{\mu'}\right), \\ u_2(\bar{x}) &= U\left(\frac{\lambda p}{\mu'}, \frac{\lambda p-\mu+\mu'}{\mu'}, -\frac{\lambda q c_1 \bar{x}}{\mu'}\right), \end{aligned} \tag{3.18}$$

where, writing $a=\lambda p/\mu'$, $b=(\lambda p-\mu+\mu')/\mu'$ and $z=-(\lambda q c_1 \bar{x}/\mu')$, $M(a, b, z)$ and $U(a, b, z)$ are given by

$$\begin{aligned} M(a, b, z) = 1 &+ \frac{az}{b} + \frac{a(a+1)}{b(b+1)}\frac{z^2}{2!} + \cdots \\ &+ \frac{a(a+1)(a+2)\cdots(a+n-1)z^n}{b(b+1)(b+2)\cdots(b+n-1)n!} + \cdots \end{aligned} \tag{3.19}$$

and

$$U(a, b, z) = \frac{\pi}{\sin \pi b}\left[\frac{M(a, b, z)}{\Gamma(1+a-b)\Gamma(b)} - z^{1-b}\frac{M(1+a-b, 2-b, z)}{\Gamma(a)\Gamma(2-b)}\right] \tag{3.20}$$

(Slater [8]). Then the solution of the Riccati equation (3.9) is

$$\bar{y} = r-1 = -\frac{u_2'(\bar{x}) + c_2 u_1'(\bar{x})}{h(\bar{x})(c_2 u_1(\bar{x}) + u_2(\bar{x}))} \quad \text{(Murphy [6])} \tag{3.21}$$

i.e.,

$$c_2 = -\frac{\bar{y}h(\bar{x})u_2(\bar{x}) + u_2'(\bar{x})}{\bar{y}h(\bar{x})u_1(\bar{x}) + u_1'(\bar{x})}, \tag{3.22}$$

where $c_2$ is another constant.

Now putting back $\bar{y} = r-1$, $\bar{x} = e^{\mu' t}$ and substituting $h(\bar{x})$ from (3.10) in (3.22) gives

$$\frac{\mu' e^{\mu' t} u_2' - \lambda p(r-1)}{\lambda p(r-1)u_1 - \mu' e^{\mu' t} u_1'} = \text{constant} \tag{3.23}$$

where $u_1$ and $u_2$ stand for the functions $u_1(\bar{x})$ and $u_2(\bar{x})$ respectively. Hence from (3.5) and (3.23) we get

$$F(r, s, t) = \emptyset\left[(s-1)\,e^{-\mu' t}, \frac{\mu' e^{\mu' t} u_2' - \lambda p(r-1)u_2}{\lambda p(r-1)u_1 - \mu' e^{\mu' t} u_1'}\right]. \tag{3.24}$$

At $t=0$, let $x=k_1=1$ and $y=k_2=0$, then

$$F(r, s, 0) = r = \emptyset\left[(s-1), \frac{\mu' u_2^{0'} - \lambda p(r-1)u_2^0}{\lambda p(r-1)u_1^0 - \mu' u_1^{0'}}\right]. \tag{3.25}$$

Put $s-1=w$ and $(\mu' u_2^{0'}-\lambda p(r-1)u_2^0)/(\lambda p(r-1)u_1^0-\mu' u_1^{0'})=V$, i.e., $s=1+w$ and

$$r=\frac{\mu' u_2^{0'}+\lambda p u_2^0+V(\lambda p u_1^0+\mu' u_1^{0'})}{\lambda p(u_2^0+u_1^0 V)};$$

then

$$\emptyset(w, V)=\frac{\mu' u_2^{0'}+\lambda p u_2^0+V(\lambda p u_1^0+\mu' u_1^{0'})}{\lambda p(u_2^0+u_1^0 V)}=1+\frac{\mu'(u_2^{0'}+u_1^{0'} V)}{\lambda p(u_2^0+u_1^0 V)}.$$

Hence,

$$\begin{aligned}
F(r, s, t) &= 1+\frac{\mu'}{\lambda p}\frac{u_2^{0'}+u_1^{0'}\left[\dfrac{\mu' u_2' \mathrm{e}^{\mu' t}-\lambda p(r-1)u_2}{\lambda p(r-1)u_1-\mu' u_1' \mathrm{e}^{\mu' t}}\right]}{u_2^0+u_1^0\left[\dfrac{\mu' u_2 \mathrm{e}^{\mu' t}-\lambda p(r-1)u_2}{\lambda p(r-1)u_1-\mu' u_1' \mathrm{e}^{\mu' t}}\right]} \\
&= 1+\frac{\mu'}{\lambda p}\frac{u_2^{0'}A-u_1^{0'}B}{u_2^0 A-u_1^0 B}
\end{aligned} \tag{3.26}$$

where $A=\lambda p(r-1)u_1-\mu' u_1' \mathrm{e}^{\mu' t}$ and $B=\lambda p(r-1)u_2-\mu' u_2' \mathrm{e}^{\mu' t}$.

Thus we get the explicit expression of the generating function $F(r, s, t)$ of the joint probability distribution $P_{x,y}(t)$.

The determination of the joint probability distribution $P_{x,y}(t)$ from (3.26) would be very tedious. Note also that the interesting moments of both $X(t)$ and $Y(t)$ can be obtained much more simply by other methods, and that the probability generating function for $X(t)$ is well-known. Formula (3.26) only really helps us with the probability generating function for $Y(t)$.

In the next section we shall utilize the expression (3.26) to obtain the marginal distribution and moments of $X(t)$ and $Y(t)$.

## 4. The marginal distribution $P_x(t)$

From the definition, the generating function $F(r, t)$ of the marginal distribution $P_x(t)$ is given by

$$F(r, t)=F(r, s, t)|_{s=1}. \tag{4.1}$$

Now, as $s\to 1$ (i.e., $z\to 0$), we have from (3.18), (3.19) and (3.20),

$$\begin{aligned}
u_1 &= 1, \\
u_1' &= \frac{a}{b}\left(-\frac{\lambda q c_1}{\mu'}\right) \quad \text{for } \lambda p\neq\mu, \\
&= a\left(-\frac{\lambda q c_1}{\mu'}\right) \quad \text{for } \lambda p=\mu \qquad (\text{since } b=1 \text{ for } \lambda p=\mu),
\end{aligned}$$

$$u_2 = \frac{\pi}{\sin \pi b}\left[\frac{1}{\Gamma(1+a-b)\Gamma(b)} - \frac{z^{1-b}}{\Gamma(a)\Gamma(2-b)}\right] \quad \text{for } \lambda p \neq \mu,$$

(Slater [8])

$$= -\frac{1}{\Gamma(a)}[\log z + \psi(a)] \quad \text{for } \lambda p = \mu \tag{4.2}$$

$$u_2' = \frac{\pi}{\sin \pi b}\left[\frac{(\lambda q c_1/\mu')(1-b)z^{-b}}{\Gamma(a)\Gamma(2-b)}\right] \quad \text{for } \lambda p \neq \mu,$$

$$= -\frac{1}{z\Gamma(a)}\left(-\frac{\lambda q c_1}{\mu'}\right) \quad \text{for } \lambda p = \mu,$$

and for $u_i^0$ and $u_i^{0'}$, similar expressions replacing $z$ by $z_0$. Also, remembering from (3.5) and (3.18) that

$$z = -\frac{\lambda q c_1 x}{\mu'} = -\frac{\lambda q}{\mu'}(s-1) \quad \text{and} \quad z_0 = -\frac{\lambda q c_1}{\mu'} = -\frac{\lambda q}{\mu'}(s-1)\,\mathrm{e}^{-\mu' t} \tag{4.3}$$

and setting $s = 1$ we substitute (4.2) in (3.26) to obtain directly the well known p.g.f.'s for the linear birth-and-death process $X(t)$ with birth rate $\lambda p$ and death rate $\mu$, in both the cases $\lambda p \neq \mu$ and $\lambda p = \mu$ as given in (2.8) and (2.10), leading to the probability distributions (2.7) and (2.9) respectively.

To obtain expressions for the mean and variance of $X(t)$ we note that

$$E[X(t)] = \left.\frac{\partial F(r, t)}{\partial r}\right|_{r=1} = \mathrm{e}^{(\lambda p - \mu)t}, \tag{4.4}$$

and

$$\sigma_x^2 = \left.\frac{\partial^2 F}{\partial r^2}\right|_{r=1} + \left.\frac{\partial F}{\partial r}\right|_{r=1} - \left[\left.\frac{\partial F}{\partial r}\right|_{r=1}\right]^2 = \frac{\lambda p + \mu}{\lambda p - \mu}\,\mathrm{e}^{(\lambda p - \mu)t}(\mathrm{e}^{(\lambda p - \mu)t} - 1). \tag{4.5}$$

All the other higher moments of $X(t)$ can be obtained by similar calculations.

## 5. The marginal distribution $P_y(t)$

As before the generating function $F(s, t)$ of the marginal distribution $P_y(t)$ can be obtained directly from the generating function $F(r, s, t)$ in the following way:

$$F(s, t) = F(r, s, t)|_{r=1}. \tag{5.1}$$

Hence from (3.26) and (5.1) we have

$$F(s, t) = 1 + \frac{\mu'}{\lambda p}\,\frac{u_1^{0'} u_2' - u_1' u_2^{0'}}{u_1^0 u_2' - u_1' u_2^0}. \tag{5.2}$$

(5.2) is the explicit expression for the p.g.f. of the marginal distribution $P_y(t)$.

For the sake of simplification we shall consider, without loss of generality, the particular case

$$\lambda p = \lambda q = \mu' \quad \text{and} \quad \mu = 0. \tag{5.3}$$

Then it can easily be found that an independent pair of solutions satisfying equation (3.17) is given by

$$u_1 = \frac{e^z - 1}{z}, \qquad u_2 = \frac{1}{z} \tag{5.4}$$

and for $u_1^0$ and $u_2^0$, similar expressions replacing $z$ by $z_0$, where $z$ and $z_0$ are given by (4.3) as before.

Hence substituting (5.3) and (5.4) in (5.2) we get after some algebraic computations,

$$F(s, t) = e^{-\mu' t}\left[\sum_n \sum_m s^{n+m}\, e^{kn}\frac{(-kn)^m}{m!} - \sum_n \sum_m s^{n+m+1}\, e^{kn}\frac{(-kn)^m}{m!}\right] \tag{5.5}$$

where $k = 1 - e^{-\mu' t}$.

Hence,

$$\begin{aligned} P_y(t) &= e^{-\mu' t}\left[\sum_{n+m=y}\sum e^{kn}\frac{(-kn)^m}{m!} - \sum_{n+m+1=y}\sum e^{kn}\frac{(-kn)^m}{m!}\right] \\ &= e^{-\mu' t}\left[\sum_{n=0}^{y} e^{kn}\frac{(-kn)^{y-n}}{(y-n)!} - \sum_{n=0}^{y-1} e^{kn}\frac{(-kn)^{y-n-1}}{(y-n-1)!}\right] \\ &= e^{-\mu' t}\left[\sum_{n=0}^{y} e^{kn}\frac{(-kn)^{y-n}}{(y-n)!} - \sum_{n=0}^{y-1} e^{kn}\frac{(y-n)(-kn)^{y-n}}{-kn(y-n)!}\right] \\ &= e^{-\mu t}\left[e^{ky} + \sum_{n=0}^{y-1} e^{kn}\frac{(-kn)^{y-n}}{(y-n)!}\left\{1 + \frac{y-n}{kn}\right\}\right] \end{aligned} \tag{5.6}$$

and

$$P_0(0) = 1. \tag{5.7}$$

Next, we shall calculate the moments of $Y(t)$ in the general case utilizing the generating function (5.2) of the probability distribution $P_y(t)$.

From (5.2) we have as before,

$$E[Y(t)] = \left.\frac{\partial F(s, t)}{\partial s}\right|_{s=1} = \frac{\mu'}{\lambda p}\frac{\partial}{\partial s}\left.\left[\frac{u_1^{0'} u_2' - u_1' u_2^{0'}}{u_1^0 u_2' - u_1' u_2^0}\right]\right|_{s=1}. \tag{5.8}$$

Using (3.18), (3.19), (3.20) and (4.3) in (5.8) we get after some computations,

$$E[y(t)] = \frac{\lambda q}{\lambda p - \mu + \mu'}[e^{(\lambda p - \mu)t} - e^{-\mu' t}]. \tag{5.9}$$

Again, differentiating (5.2) twice with respect to $s$, setting $s=1$ and by similar calculations as above we can show that the variance $\sigma_y^2$ is given by

$$\sigma_y^2=\frac{(\lambda p+\mu)\lambda^2 q^2}{(\lambda p-\mu+\mu')^2}\left\{\frac{e^{2(\lambda p-\mu)t}}{(\lambda p-\mu)}+\frac{2e^{(\lambda p-\mu+\mu')t}}{\mu'}-\frac{e^{-2\mu' t}}{\lambda p-\mu+2\mu'}\right\}$$
$$-\frac{2\lambda^2 q^2(\lambda p+\mu)\,e^{(\lambda p-\mu)t}}{\mu'(\lambda p-\mu)(\lambda p-\mu+2\mu')}+E[y(t)]. \tag{5.10}$$

We shall not give here the detailed calculations since this will be rather lengthy.

Now, we shall calculate the covariance of $X(t)$ and $Y(t)$. From the definition of probability generating function $F(r, s, t)$, we have

$$\operatorname{Cov}[X(t), Y(t)]=\left.\frac{\partial^2 F(r, s, t)}{\partial r\,\partial s}\right|_{\substack{r=1\\ s=1}}. \tag{5.11}$$

The expression (3.26) can be written as

$$F(r, s, t)=1+\frac{\mu'}{\lambda p}\left[\frac{P+Q(r-1)}{R+S(r-1)}\right] \tag{5.12}$$

where

$$\begin{aligned} P&=\mu' e^{\mu' t}(u_1^{0'}u_2'-u_1'u_2^{0'}), \qquad & Q&=\lambda p(u_1 u_2^{0'}-u_1^{0'}u_2),\\ R&=\mu' e^{\mu' t}(u_1^{0}u_2'-u_1'u_2^{0}), \qquad & S&=\lambda p(u_1 u_2^{0}-u_1^{0}u_2). \end{aligned} \tag{5.13}$$

Hence

$$\left.\frac{\partial F(r, s, t)}{\partial r}\right|_{r=1}=\frac{\mu'}{\lambda p}\left(\frac{Q}{R}-\frac{PS}{R^2}\right), \tag{5.14}$$

and

$$\left.\frac{\partial^2 F(r, s, t)}{\partial r\,\partial s}\right|_{r=1}=\frac{\mu'}{\lambda p}\left[\frac{\partial Q}{\partial s}\frac{1}{R}-\frac{Q}{R^2}\frac{\partial R}{\partial s}-\frac{S}{R^2}\frac{\partial P}{\partial s}\right.$$
$$\left.+\text{terms involving higher powers of }(s-1)\right]. \tag{5.15}$$

Using (3.19), (3.20) and (5.13) and finally setting $s=1$ in (5.15) we get, after some algebraic operation,

$$\left.\frac{\partial^2 F(r, s, t)}{\partial r\,\partial s}\right|_{\substack{r=1\\ s=1}}=\operatorname{Cov}[X(t), Y(t)]$$
$$=\frac{\lambda q(\lambda p+\mu)}{\mu'(\lambda p-\mu)(\lambda p-\mu+\mu')}[(\lambda p-\mu)\,e^{-\mu' t}+\mu'\,e^{(\lambda p-\mu)t}$$
$$-(\lambda p-\mu+\mu')]\,e^{(\lambda p-\mu)t}. \tag{5.16}$$

All other higher moments can be found out similarly extending the procedure as followed above.

## 6. Discussion

In this paper we have made a direct approach to obtain the explicit solution of the probability generating function of the joint distribution of a population of two sexes where the birth of a male depends upon the female population size, i.e., in the case where the females are marriage dominant. To the best of our knowledge no attempt has until now been made to obtain an explicit expression for the probability generating function in such cases (Gani and Tin [3a], as mentioned in the introduction, have simultaneously solved the problem by a different approach). The previous authors have, however, successfully obtained the first and second order moments of the male population size indirectly with the help of the moment generating function. But their methods do not give any idea of the probability distribution of a marriage recessive population (here the male population) which, in our opinion, is very much essential to obtain a clear knowledge of the stochastic process in question. The explicit solution of the probability generating function obtained by us provides all necessary stochastic information on the population under investigation.

Although the method of solution adopted here has been used to deal with a particular case where the females are marriage dominant this can be applied in the same way in the reverse case that is, where the males are marriage dominant and also in the case where neither males nor females are marriage dominant.

This method can also be successfully used in the studies of reversible or irreversible chemical and biochemical processes in which the production or synthesis of a species (product) depends upon the concentration of another species (source). Also in the genetic transcription process where the RNA molecules are irreversibly synthesized by the DNA molecules and the synthesis of DNA does not explicitly depend upon the RNA population, the probability distribution of RNA population can be easily found out by this method (Tapaswi and Roychoudhury).

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## References

[1] N.T.J. Bailey, The Elements of Stochastic Processes, Chapter 4, pp. 119–120 (John Wiley and Sons, Inc., New York, 1967).

[2] A.T. Bharucha-Reid, Elements of the Theory of Markov Processes and their Applications, Chapt. 2 (p. 88) and Chapt. 4 (pp. 175–179) (McGraw-Hill, New York, 1960).

[3a] J. Gani and P. Tin, A note on the two-sex population, Tech. Rept. No. 219, Department of Statistics, University of Kentucky, Kentucky, USA (1984).
[3b] J. Gani, Personal communication.
[4] L.A. Goodman, Population growth of the sexes, Biometrics 9 (1953) 212-225.
[5] D.D. Joshi, Les processus stochastiques en demographie, Publ. Inst. Statist. Univ. Paris 3 (1954) 153-177.
[6] G.M. Murphy, Ordinary Differential Equations and their Solutions, pp. 15-16 (D. Van Nostrand Co., Princeton, New Jersey, 1960).
[7] J.H. Pollard, Mathematical Models for the Growth of Human Populations, pp. 91-92 (Cambridge University Press, 1973).
[8] L.J. Slater, Confluent hypergeometric functions, in: M. Abramowitz and I.A. Stegun, eds., Handbook of Mathematical Functions (Nat. Bur. Stand., USA, 1964).
[9] P.K. Tapaswi and R.K. Roychoudhury, unpublished.